Supplemental Material

Crowdsourcing in Cyber-Physical Systems: Stochastic Optimization with Strong Stability

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Abstract—This document provides the supplemental material for our main paper titled "Crowdsourcing in Cyber-Physical Systems: Stochastic Optimization with Strong Stability". In particular, Appendix A, Appendix B, and Appendix C present the detailed proof of Lemma 1, Lemma 2, and Lemma 3, respectively. Note that we refer to the equations in the main paper by the equation numbers therein.

Index Terms—Cyber-physical systems; crowdsourcing; stochastic optimization; network strong stability.

Appendix A
Proof of Lemma 1

Note that $\forall x, y, z$ with $x \geq 0$, $0 \leq y \leq y_{max}$, $0 \leq z \leq z_{max}$, we have

$$\begin{aligned}(\max\{x-y,0\}+z)^2 &\leq x^2+y^2+z^2+2x(z-y)\\ &\leq x^2+y_{max}^2+z_{max}^2+2x(z-y). \quad \text{(A.1)}\end{aligned}$$

Due to (11), we know that $\sum_{j\in\mathcal{T}_i} f_{ij}^r(t) \leq \sum_{j\in\mathcal{T}_i}\sum_{r\in\mathcal{R}} f_{ij}^r(t) \leq \frac{1}{\alpha}\sum_{j\in\mathcal{T}_i}\sum_{m\in\mathcal{M}_i\cap\mathcal{M}_j} c_{ij}^m(t)s_{ij}^m(t)\Delta t$. Denote by c_{ij}^{max} the maximum possible link capacity of link (i,j). Since according to (1) $c_{ij}^m(t)$ depends on $d(i,j)$, P_i^m and $W^m(t)$, among which $d(i,j)$ and P_i^m are constants, then c_{ij}^{max} is determined by W^{max}, i.e., the maximum bandwidth that the channels available on link (i,j) can have. Thus, since $\sum_{m\in\mathcal{M}_j}\sum_{q\in\mathcal{T}_j^m} s_{jq}^m(t) \leq 1$ as in (9), i.e., one node can transmit to at most one neighbor on at most one band at a time, we can get $\sum_{j\in\mathcal{T}_i} f_{ij}^r(t) \leq \max_{j\in\mathcal{T}_i}\{\frac{1}{\alpha}c_{ij}^{max}\Delta t\}$. Similarly, we also have $\sum_{\{j|i\in\mathcal{T}_j\}} f_{ji}^r(t) \leq \max_{\{j|i\in\mathcal{T}_j\}}\{\frac{1}{\alpha}c_{ji}^{max}\Delta t\}$. Besides, we have $\epsilon_i^r(t) \leq O_i^{max}$ and $v_r(t) \leq A_r^{max}$. Thus, according to (5) and (A.1), we can obtain the following inequality

$$\begin{aligned}&\big(Q_i^r(t+1)\big)^2 \leq \big(Q_i^r(t)\big)^2 + \big(\max_{j\in\mathcal{T}_i}\{\frac{1}{\alpha}c_{ij}^{max}\Delta t\}\\ &+O_i^{max}\big)^2 + \big(\max_{\{j|i\in\mathcal{T}_j\}}\{\frac{1}{\alpha}c_{ji}^{max}\Delta t\} + A_r^{max}\cdot 1_{i=s_r}\big)^2\\ &+2Q_i^r(t)\cdot\big(\sum_{\{j|i\in\mathcal{T}_j\}} f_{ji}^r(t) + v_r(t)\cdot 1_{i=s_r}\\ &-\sum_{j\in\mathcal{T}_i} f_{ij}^r(t) - \epsilon_i^r(t)\big). \quad \text{(A.2)}\end{aligned}$$

Applying this inequality to the drift-plus-penalty function, we have

$$\begin{aligned}&\Delta(\mathbf{Q}(t)) - V\mathbb{E}[\sum_{r\in\mathcal{R}} v_r(t)|\mathbf{Q}(t)]\\ =&\frac{1}{2}\mathbb{E}\big[\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\big[\big(Q_i^r(t+1)\big)^2 - \big(Q_i^r(t)\big)^2\big]|\mathbf{Q}(t)\big]\\ &-V\mathbb{E}\big[\sum_{r\in\mathcal{R}} v_r(t)|\mathbf{Q}(t)\big]\\ \leq&\frac{1}{2}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\big(\max_{j\in\mathcal{T}_i}\{\frac{1}{\alpha}c_{ij}^{max}\Delta t\} + O_i^{max}\big)^2\\ &+\frac{1}{2}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\big(\max_{\{j|i\in\mathcal{T}_j\}}\{\frac{1}{\alpha}c_{ji}^{max}\Delta t\} + A_r^{max}\cdot 1_{i=s_r}\big)^2\\ &+\mathbb{E}\big[\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} Q_i^r(t)\big(\sum_{\{j|i\in\mathcal{T}_j\}} f_{ji}^r(t) + v_r(t)\cdot 1_{i=s_r}\\ &-\sum_{j\in\mathcal{T}_i} f_{ij}^r(t) - \epsilon_i^r(t)\big)|\mathbf{Q}(t)\big] - V\mathbb{E}\big[\sum_{r\in\mathcal{R}} v_r(t)|\mathbf{Q}(t)\big]\end{aligned}$$

Thus, Lemma 1 directly follows.

Appendix B
Proof of Lemma 2

According to (11) and (17), we get $\sum_{r\in\mathcal{R}} f_{ij}^r(t) \leq \frac{1}{\alpha}c_{ij}^{max}\Delta t$. Thus, according to the queueing law (16) and the inequality (A.1), we can have

$$\begin{aligned}&(Y_{ij}(t+1))^2 \leq (Y_{ij}(t))^2 + 2(\frac{\sigma}{\alpha}c_{ij}^{max}\Delta t)^2\\ &+2Y_{ij}(t)(\sum_{r\in\mathcal{R}}\sigma f_{ij}^r(t) - \frac{\sigma}{\alpha}\sum_{m\in\mathcal{M}_i\cap\mathcal{M}_j} c_{ij}^m(t)s_{ij}^m(t)\Delta t).\end{aligned} \quad \text{(B.1)}$$

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According to (A.2) and (B.1), we can obtain

$$\begin{aligned}
&\Delta(\mathbf{\Theta}(t)) - V\mathbb{E}\big[\sum_{r\in\mathcal{R}} v_r(t)|\mathbf{\Theta}(t)\big] \\
\leq&\frac{1}{2}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\big(\max_{j\in\mathcal{T}_i}\{\frac{1}{\alpha}c_{ij}^{max}\Delta t\} + O_i^{max}\big)^2 \\
&+\frac{1}{2}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\big(\max_{\{j|i\in\mathcal{T}_j\}}\{\frac{1}{\alpha}c_{ji}^{max}\Delta t\} + A_r^{max}\cdot 1_{i=s_r}\big)^2 \\
&+\frac{1}{2}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{T}_i}2\big(\frac{\sigma}{\alpha}c_{ij}^{max}\Delta t\big)^2 \\
&+\mathbb{E}\big[\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}Q_i^r(t)\big(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^r(t) + v_r(t)\cdot 1_{i=s_r} \\
&-\sum_{j\in\mathcal{T}_i}f_{ij}^r(t) - \epsilon_i^r(t)\big)|\mathbf{Q}(t)\big] \\
&+\mathbb{E}\big[\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{T}_i}Y_{ij}(t)\big(\sum_{r\in\mathcal{R}}\sigma f_{ij}^r(t) - \\
&\frac{\sigma}{\alpha}\sum_{m\in\mathcal{M}_i\cap\mathcal{M}_j}c_{ij}^m(t)s_{ij}^m(t)\Delta t\big)|\mathbf{Y}(t)\big] - V\mathbb{E}\big[\sum_{r\in\mathcal{R}}v_r(t)|\mathbf{\Theta}(t)\big].
\end{aligned}$$

Thus, Lemma 2 directly follows.

Appendix C
Proof of Lemma 3

Denote the optimal amount of CPS services that maximizes P1 and minimizes P2 by $v_r^*(t)$ and $\hat{v}_r^*(t)$, respectively. Thus, according to (14), we can know that

$$\begin{aligned}
\Delta(\mathbf{Q}(t)) - V\mathbb{E}[\sum_{r\in\mathcal{R}}\hat{v}_r^*(t)|\mathbf{Q}(t)] \leq& B + \Psi(t)|_{v_r(t)=\hat{v}_r^*(t)} \\
\leq& B + \Psi(t)|_{v_r(t)=v_r^*(t)},
\end{aligned}$$

since the optimal solution to P1 is a feasible solution to P2. As a result, we get

$$\begin{aligned}
&\Delta(\mathbf{Q}(t)) - V\mathbb{E}[\sum_{r\in\mathcal{R}}\hat{v}_r^*(t)|\mathbf{Q}(t)] \\
\leq& B - V\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)|\mathbf{Q}(t)] + \mathbb{E}\big[\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}Q_i^r(t)\cdot \\
&(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t))|\mathbf{Q}(t)\big] \\
=& B - V\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)] + \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}Q_i^r(t)\mathbb{E}\big[\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) \\
&+v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t)\big] \\
=& B - V\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)] \\
&+\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}Q_i^r(t)\big[\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + v_r^*(t)\cdot 1_{i=s_r} \\
&-\sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t))\big].
\end{aligned}$$

The second step is because the optimal solution to P1 is independent of the system queue size $\mathbf{Q}(t)$. The third step is due to the strong law of large numbers: If $\{a(t)\}_{t=0}^{\infty}$ are i.i.d. random variables, we have $\Pr(\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}a(t) = \mathbb{E}\{a(t)\}) = 1$. Particularly, since the optimal solution to P1 is obtained based on the available spectrum bandwidths and computing resources, which are assumed to be i.i.d. random processes as mentioned before, then $\{\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t)\}_{t=0}^{\infty}$ are i.i.d. random variables as well and their time average converges almost surely to the expected value.

Taking expectation of the above inequality, we have

$$\begin{aligned}
&\mathbb{E}[L(\mathbf{Q}(t+1))] - \mathbb{E}[L(\mathbf{Q}(t))] - V\cdot\mathbb{E}[\sum_{r\in\mathcal{R}}\hat{v}_r^*(t)] \\
\leq& B - V\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)] \\
&+\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\mathbb{E}[Q_i^r(t)]\big[\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + \\
&v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t))\big] \\
\leq& B - V\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)].
\end{aligned}$$

We are able to take expectation of $Q_i^r(t)$ and $\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t)$ separately because the optimal solution to P1 is independent of the system queue size. Meanwhile, we have $\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^{r*}(t) + v_r^*(t)\cdot 1_{i=s_r} - \sum_{j\in\mathcal{T}_i}f_{ij}^{r*}(t) - \epsilon_i^{r*}(t)) \leq 0$. The reason is that for each $Q_i^r(t)$, we have

$$\begin{aligned}
&\sum_{j\in\mathcal{T}_i}f_{ij}^r(t) + \epsilon_i^r(t) - \big(\sum_{\{j|i\in\mathcal{T}_j\}}f_{ji}^r(t) + v_r(t)\cdot 1_{i=s_r}\big) \\
\leq& \sum_{j\in\mathcal{T}_i}f_{ij}^r(t) + \epsilon_i^r(t) \leq \max_{j\in\mathcal{T}_i}\{\frac{1}{\alpha}c_{ij}^{max}\Delta t\} + O_i^{max},
\end{aligned}$$

which is a finite constant. Since in P1, $\mathbf{Q}(t)$ is strongly stable, we can ensure its rate stability according to Theorem 2.

Summing the above over $t \in \{0, 1, 2, ..., T-1\}$ for any positive integer T yields

$$\begin{aligned}
&\mathbb{E}[L(\mathbf{Q}(T))] - \mathbb{E}[L(\mathbf{Q}(0))] - V\sum_{t=0}^{T-1}\mathbb{E}[\sum_{r\in\mathcal{R}}\hat{v}_r^*(t)] \\
&\leq TB - V\sum_{t=0}^{T-1}\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)].
\end{aligned}$$

Since $L(\mathbf{Q}(0)) = 0$ and $L(\mathbf{Q}(T)) \geq 0$, we have

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\sum_{r\in\mathcal{R}}\hat{v}_r^*(t)] \geq \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\sum_{r\in\mathcal{R}}v_r^*(t)] - \frac{B}{V}.$$

Taking limitation as $T \to \infty$ leads to

$$\psi_{P2}^* \geq \psi_{P1}^* - \frac{B}{V}.$$